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A method to derive approximate explicit solutions for structural mechanics problems

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Abstract

The availability of explicit solutions, i.e. analytical relationships between the structural response and the design variables, allows a more direct and plain treatment of several structural problems. This paper is devoted to derive approximate explicit solutions in the framework of linear static analysis of finite element modeled structures with a given layout (fixed node positions). The proposed procedure is based on a factorization of the element stiffness matrix following the *unimodal components* concept, which allows a non-conventional assembly of the global stiffness matrix. The exact inversion of that matrix is a trivial task for the case of statically determinate structures, structures with few redundancies or few design variables. An approximate inverse of the stiffness matrix is herein derived for more general structural problems by resorting to the Sherman–Morrison–Woodbury formula.

Keywords: Sherman–Morrison–Woodbury formula; Explicit solutions; Parametric solutions

1. Introduction

Standard structural analysis tools are devoted to numerical evaluation of the system response due to external loads for given geometry and material properties. However, in many cases one may want to investigate the behavior of companion structures that are similar to an initial design structure or to fix the structural parameters so to meet some given requirements. Such instances occur in reanalysis, Monte Carlo simulation, probabilistic analysis, system identification, structural optimization, damage detection, sensitivity analysis, etc. The treatment of these problems requires to find the response of several modified structures which differ, for example, for element properties, geometry, boundary conditions and loading. If the problem dimension is large, the fresh analysis of each modified structure would require a large number of simultaneous equations to be reformed and solved, which may lead to heavy computational cost. Several techniques have been developed with the aim of evaluating the structural response for the modified structures exploiting the analysis of the initial design structure such that the computational effort is less than that required by a fresh analysis. For

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static setting, a survey of these procedures is given in the review paper by Abu Kassim and Topping (1987), where the analysis algorithms are divided into two general categories: direct (i.e. exact methods) and approximate methods. Direct methods provide exact explicit solutions which are efficient if the number of modified elements is small; approximate methods are more general but usually in order to achieve accuracy they resort to iterative algorithms which may considerably reduce, even nullify, convenience over a fresh analysis (Kavlie and Powell, 1971; Topping and Abu Kassim, 1987).

Recently, several attempts have been made in order to build approximate procedures which exploit the results produced by direct methods so to construct approximate analytical expressions of the structural response, explicit in the element properties. The effort to exploit exact explicit solutions as a basis to produce approximate models appears appropriate. In fact, a procedure able to furnish approximate explicit solutions, which at least are exact for a class of structural settings, is candidate to perform well for more complex scenarios, and to produce accurate results with no need of iterations. For example, the Combined Approximation (Garcelon et al., 2000), one of the most accurate methods recently developed, can be tuned so to give exact results for truss structures with modifications made to the axial stiffness of a given number of members (Kirsch and Liu, 1995). The Reciprocal Approximation (Schmit and Farhi, 1974), which expresses the nodal displacements as linear Taylor polynomials in terms of the member flexibility, is exact for statically determinate structures and quite accurate for redundant structures (Fuchs and Shabtay, 2000). The Response Surface Method using ad hoc rational polynomials as performance functions (Falsone and Impollonia, 2004) produces exact explicit relationships for finite element modeled structures when a single element property is treated as a parameter; very accurate predictions are obtained for more general cases.

The availability of exact explicit relationships between the response and the variable structural parameters is, then, crucial for devising accurate approximate explicit solutions. For the simplest case of linear structures under static loads, to find an exact explicit solution is equivalent to derive the exact inverse of the parameterized stiffness matrix. To quote Fuchs, who devised very interesting results on this matter: "the quest for the explicit inverse of the stiffness matrix in structural theory bears some resemblance to the quest for Eldorado (the golden one) by the Spanish conquistadores" (Fuchs, 1992a).

The exact explicit solution for statically determinate structures has been obtained by several authors following different paths. For example, in the framework of uncertain structures, Falsone and Impollonia (2002) employed a coordinate transformation based on the solution of eigenvalue problems; Elishakoff and Ren (2003) inverted the parametric stiffness matrix which was assembled in a non-conventional way; Di Paola (2004) applied the Virtual Distorsion Method.

The case of statically indeterminate structures is quite more involved due to non-linear dependence of nodal displacements on element flexibilities (a linear relationship is, instead, assured for statically determinate structures). Exact explicit solutions have been derived only for simple structures: single beam (Elishakoff et al., 1997; Elishakoff and Ren, 2003; Falsone and Sofi, 2004), structures possessing a single element with variable properties (Falsone and Impollonia, 2002, 2004), structures comprising few elements and few redundancies (Fuchs, 1992b).

Fuchs (1992a) highlighted that exact explicit solutions for more complex structures are theoretically obtainable but they are practically unusable for the immense number of terms involved. Then, in this case, accurate approximate solutions would be much more useful. Fuchs and Maslovitz (1992) developed approximate explicit relationships between the axial stress of truss structures and the element stiffness by retaining only a fraction of terms and enforcing compatibility of deformation at preselected points. Falsone and Impollonia (2004) proposed a class of approximate solutions for the nodal displacements of general structures by including cross-terms to increase accuracy.

The present contribution is aimed at obtaining approximate analytical solutions for the nodal displacements of linear structures with constant layout (fixed node positions) under static loads, by determining an approximate inverse of the parametric stiffness matrix. The solutions are explicit in the element parameters: cross-section properties and material constants. The procedure is based on a factorization of the element stiffness matrix, following the *unimodal components* concept (Fuchs, 1991, 1997), which allows to disassembly the generic finite element into superposition of more unimodal components, i.e. simple sub-elements possessing only one natural mode. Basically, finite element disassembly consists of representing finite element matrices as a matrix product, where topology contributions are isolated from constitutive law (Hemez and Pagnacco, 2000). The general technique presented by Rong and Lü (1994) can be applied for generating sub-elements of a given finite element. Then, the proposed procedure requires a non-conventional assemblage of the global stiffness matrix, which is based on the use of unimodal components and allows a truss-like formulation. The global stiffness matrix is built up as the product of the statics matrix, the uncoupled constitutive matrix and the kinematics matrix, each unimodal component explicitly appearing in the diagonal constitutive matrix. A similar assemblage of the stiffness matrix was also used by Elishakoff et al. (1997) for straight beams, by Doebling et al. (1998) in order to parameterize the global stiffness matrix for solving structural identification problems and by Hemez and Pagnacco (2000) for devising an efficient numerical solver.

The non-conventional assemblage of the stiffness matrix allows to obtain straightforwardly its inverse for any statically determinate structure. For structures with few redundancies or few element properties treated as parameters, the Sherman–Morrison–Woodbury (SMW) formula (Henderson and Searle, 1981; Hager, 1989) is resorted to, so that an exact inverse of the stiffness matrix is still possible. As pointed out by Akgün et al. (2001), the SMW is a very general tool for evaluating inexpensively the response of structures with low-rank modifications and most of the structural modification techniques are related to this formula.

The SMW formula is exploited herein to get the approximate inverse of the stiffness matrix for the case of a generic structure with many redundancies and parameters (high-rank modifications). A first solution is obtained by simply adding up the contributions due to each unimodal component, which leads to satisfactory results only for small parameter fluctuations. A more refined and effective approximation is proposed through cross-terms inclusion which accounts for the interaction effects between fluctuations of the various components. The accuracy of the resulting explicit solution is remarkable as evidenced by several numerical applications, where the approximate nodal displacements and internal stresses of finite element modeled structures are compared with the exact ones.

2. Element stiffness matrix

Let us consider the equilibrium equation of a generic finite element

$$\mathbf{K}^{(e)}(\boldsymbol{\alpha}^{(e)})\mathbf{u}^{(e)} = \mathbf{F}^{(e)} \tag{1}$$

where $\mathbf{K}^{(e)}(\boldsymbol{\alpha}^{(e)})$ is the stiffness matrix, $\mathbf{u}^{(e)}$ and $\mathbf{F}^{(e)}$ are the vectors of nodal displacements and forces. The stiffness matrix is an explicit function of the element cross-section properties, the element material properties and the element geometry, which are arranged in the vector $\boldsymbol{\alpha}^{(e)}$. For example, the area $A^{(e)}$ and the moment of inertia $I^{(e)}$ of a beam element or the thickness $t^{(e)}$ of a 2D element represent cross-section properties, whereas the Young modulus $E^{(e)}$, the shear modulus $G^{(e)}$ or the Poisson's ratio $\nu^{(e)}$ correspond to material properties. Both the cross-section properties and the material properties of the elements are let to vary and play the role of basic parameters. The element geometry, i.e. the length $l^{(e)}$ of a 1D element or the shape and the surface area $S^{(e)}$ of a 2D element will not be considered herein as parameters since the position of nodes is supposed to be fixed. The proposed method starts from the following factorization of the element stiffness matrix:

$$\mathbf{K}^{(e)}(\boldsymbol{\alpha}^{(e)}) = \boldsymbol{\Phi}^{(e)}\boldsymbol{\Lambda}^{(e)}(\boldsymbol{\alpha}^{(e)})\boldsymbol{\Phi}^{(e)^{\mathrm{T}}} \tag{2}$$

being $\boldsymbol{\Lambda}^{(e)}(\boldsymbol{\alpha}^{(e)})$ a diagonal parametric matrix,

$$\boldsymbol{\Lambda}^{(e)}(\boldsymbol{\alpha}^{(e)}) = \mathrm{diag}[\boldsymbol{\alpha}^{(e)}] \tag{3}$$

whereas $\boldsymbol{\Phi}^{(e)}$ is a rectangular numerical matrix, with the number of rows exceeding that of the columns, which is not affected by the element cross-section and material properties. Eq. (2) can be easily derived for several types of elements (see Rong and Lü, 1994): some examples are given in the following.

(a) Truss element

For the two node truss element, the displacement and force vectors collect the longitudinal nodal displacements u_i and the axial forces N_i $(i = 1, 2)$

$$\mathbf{u}^{(e)} = [u_1 \quad u_2]^{\mathrm{T}}; \quad \mathbf{F}^{(e)} = [N_1 \quad N_2]^{\mathrm{T}}. \tag{4}$$

The matrix $\mathbf{\Phi}^{(e)}$ reduces to a vector and the matrix $\mathbf{\Lambda}^{(e)}(\boldsymbol{\alpha}^{(e)})$ to a scalar so that just one basic parameter $\alpha^{(e)}$ is present

$$\mathbf{\Phi}^{(e)} = \frac{1}{\sqrt{2}}[-1 \quad 1]^{\mathrm{T}}; \qquad \mathbf{\Lambda}^{(e)}(\boldsymbol{\alpha}^{(e)}) = \alpha^{(e)}; \quad \alpha^{(e)} = \frac{2E^{(e)}A^{(e)}}{l^{(e)}}. \tag{5}$$

(b) Frame element

For the case of a two node beam element with three degrees of freedom at each node, one can assume (Fuchs, 1997)

$$\mathbf{u}^{(e)} = \begin{bmatrix} u_1 & w_1 & l^{(e)}\varphi_1 & u_2 & w_2 & l^{(e)}\varphi_2 \end{bmatrix}^{\mathrm{T}}; \quad \mathbf{F}^{(e)} = \begin{bmatrix} N_1 & V_1 & \frac{M_1}{l^{(e)}} & N_2 & V_2 & \frac{M_2}{l^{(e)}} \end{bmatrix}^{\mathrm{T}} \tag{6}$$

where u_i, w_i and φ_i $(i = 1,2)$ are the longitudinal, transversal and rotational nodal degrees of freedom in the frame reference system, and N_i, V_i and M_i $(i = 1,2)$ stand, respectively, for axial and transversal forces and bending moment at the nodes.

Accordingly,

$$\mathbf{\Phi}^{(e)} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}^{\mathrm{T}}; \quad \mathbf{\Lambda}^{(e)}(\boldsymbol{\alpha}^{(e)}) = \mathrm{diag}\begin{bmatrix} \alpha_1^{(e)} & \alpha_2^{(e)} & \alpha_3^{(e)} \end{bmatrix} \tag{7}$$

where the three basic parameters read

$$\alpha_1^{(e)} = \frac{2E^{(e)}A^{(e)}}{l^{(e)}}; \quad \alpha_2^{(e)} = \frac{2E^{(e)}I^{(e)}}{l^{(e)^3}}; \quad \alpha_3^{(e)} = \frac{30\beta^{(e)}E^{(e)}I^{(e)}}{l^{(e)^3}}. \tag{8}$$

In Eq. (8) $\beta^{(e)}$ is a positive scalar $\leqslant 1$

$$\beta^{(e)} = \frac{1}{1 + 12\frac{E^{(e)}I^{(e)}}{G^{(e)}A_s^{(e)}l^{(e)^2}}} \tag{9}$$

where $A_s^{(e)}$ is the cross-section area for shear. For infinitely stiff beams in shear, $\beta^{(e)} = 1$ and Eq. (2) yields the stiffness matrix for the Euler frame element.

Both for the truss and frame elements, $\mathbf{\Lambda}^{(e)}(\boldsymbol{\alpha}^{(e)})$ and $\mathbf{\Phi}^{(e)}$ collect, respectively, the non-zero eigenvalues and the related eigenvectors (natural modes) of the stiffness matrix expressed in the local coordinate system. Namely, the matrix factorization in Eq. (2) represents a spectral decomposition.

The assemblage of the global stiffness matrix requires a coordinate transformation from the local coordinate system to the global one. This is produced through the transformation matrix $\mathbf{L}$ which accounts for the rotation from one reference system to the other. The element stiffness matrix referenced to the global system is also factorized as in Eq. (2)

$$\mathbf{K}_g^{(e)}(\boldsymbol{\alpha}^{(e)}) = \mathbf{\Phi}_g^{(e)}\mathbf{\Lambda}^{(e)}(\boldsymbol{\alpha}^{(e)})\mathbf{\Phi}_g^{(e)^{\mathrm{T}}} \tag{10}$$

where the matrix $\mathbf{\Phi}_g^{(e)}$ is obtained by projecting into the global frame system the eigenvectors matrix $\mathbf{\Phi}^{(e)}$ given in Eq. (5) or (7)

$$\mathbf{\Phi}_g^{(e)} = \mathbf{L}^{\mathrm{T}}\mathbf{\Phi}^{(e)}. \tag{11}$$

Hereinafter the subscript g will be omitted so that Eq. (2) will also be adopted in the global coordinate system.

(c) 2D constant strain triangle

The 2D constant strain triangular element has two translational degrees of freedom at each node so that

$$\mathbf{u}^{(e)} = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3]^{\mathrm{T}}; \quad \mathbf{F}^{(e)} = [X_1 \quad Y_1 \quad X_2 \quad Y_2 \quad X_3 \quad Y_3]^{\mathrm{T}} \tag{12}$$

where u_i, v_i and X_i, Y_i $(i = 1,2,3)$, respectively, are the displacements and forces at the node of coordinates (x_i, y_i) in the global reference system. According to the isoparametric representation, the matrix $\mathbf{\Phi}^{(e)}$ can be written as

$$\mathbf{\Phi}^{(e)} = \mathbf{B}^{(e)}\mathbf{\Psi}^{(e)} \tag{13}$$

where $\mathbf{B}^{(e)}$ is the element strain–displacement matrix, relating the six nodal displacements to the three strains components ε_x, ε_y, γ_{xy}. Using the notation $x_{ij} = x_i - x_j$, $y_{ij} = y_i - y_j$, the matrix $\mathbf{B}^{(e)}$ reads

$$\mathbf{B}^{(e)} = \frac{1}{x_{13}y_{23} - x_{23}y_{13}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{21} \end{bmatrix}^{\mathrm{T}}. \tag{14}$$

The matrices $\mathbf{\Psi}^{(e)}$ and $\mathbf{\Lambda}^{(e)}(\boldsymbol{\alpha}^{(e)})$ collect, respectively, the eigenvectors and the eigenvalues of the material property matrix which relates the stress vector $\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \sigma_y & \tau_{xy} \end{bmatrix}^{\mathrm{T}}$ to the strain vector $\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \gamma_{xy} \end{bmatrix}^{\mathrm{T}}$. Then, either for a plane stress or a plane strain problem

$$\mathbf{\Psi}^{(e)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}; \quad \mathbf{\Lambda}^{(e)}(\boldsymbol{\alpha}^{(e)}) = \mathrm{diag}\begin{bmatrix} \alpha_1^{(e)} & \alpha_2^{(e)} & \alpha_3^{(e)} \end{bmatrix} \tag{15}$$

where

$$\alpha_1^{(e)} = S^{(e)} \frac{E^{(e)}t^{(e)}}{2(1+\nu^{(e)})}; \quad \alpha_2^{(e)} = S^{(e)} \frac{E^{(e)}t^{(e)}}{1+\nu^{(e)}} \tag{16}$$

The third parameter $\alpha_3^{(e)}$ differs from one problem to the other; for the plane strain problem it is given by

$$\alpha_3^{(e)} = S^{(e)} \frac{E^{(e)}t^{(e)}}{1-\nu^{(e)}-2\nu^{(e)^2}} \tag{17}$$

whereas for the plane stress problem

$$\alpha_3^{(e)} = S^{(e)} \frac{E^{(e)}t^{(e)}}{1-\nu^{(e)}}. \tag{18}$$

2.1. Unimodal components

The stiffness matrix factorization given by Eq. (2) can be interpreted as a decomposition of the element stiffness matrix into its *unimodal components* (Fuchs, 1991, 1997), each of which carries a single generalized stress component and contrasts a single generalized strain. The overall element properties are reconstituted by superimposition of the unimodal components acting in parallel. Such a reasoning can be best evidenced by rewriting Eq. (2) as follows:

$$\mathbf{K}^{(e)}(\boldsymbol{\alpha}^{(e)}) = \sum_{i=1}^{N_\alpha^{(e)}} \alpha_i^{(e)}\boldsymbol{\phi}_i^{(e)}\boldsymbol{\phi}_i^{(e)^{\mathrm{T}}} = \sum_{i=1}^{N_\alpha^{(e)}} \mathbf{k}_i^{(e)}(\alpha_i^{(e)}) \tag{19}$$

where $\boldsymbol{\phi}_i^{(e)}$ are the columns of matrix $\mathbf{\Phi}^{(e)}$ and $N_\alpha^{(e)}$, equal to the dimension of the matrix $\mathbf{\Lambda}^{(e)}(\boldsymbol{\alpha}^{(e)})$, is the number of the unimodal components in the finite element. For the truss and frame element, the generalized stress and strain components are given by $\mathbf{\Phi}^{(e)^{\mathrm{T}}}\mathbf{F}^{(e)}$ and $\mathbf{\Phi}^{(e)^{\mathrm{T}}}\mathbf{u}^{(e)}$, respectively. Then, the truss element is unimodal whereas the frame element is three-modal. In particular, as shown in Fig. 1, the first unimodal component of a frame element carries axial load only, the second is stressed by a constant bending moment and the third by an antisymmetric bending moment.

Also the 2D triangle is three-modal, being $\mathbf{\Psi}^{(e)^{\mathrm{T}}}\boldsymbol{\sigma}^{(e)}$ and $\mathbf{\Psi}^{(e)^{\mathrm{T}}}\boldsymbol{\varepsilon}^{(e)}$ the generalized stress and strain components, respectively. Among the three sub-elements acting in parallel one is deputized to absorb only the shear

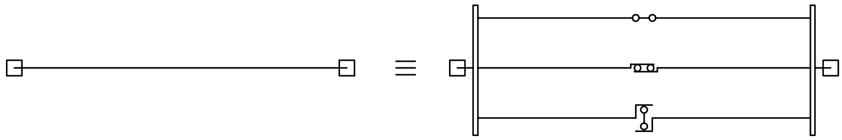

Fig. 1. Unimodal components of frame element.

stress τ_{xy}; another one will be stressed by normal stresses σ_x and σ_y with opposite values; in the last sub-element the normal stresses σ_x and σ_y will take the same value.

3. Stiffness matrix assemblage

When the element stiffness matrix in the global frame is factorized as in Eq. (2), the global stiffness matrix is amenable to a non-conventional assemblage, which evidences the contribution of each unimodal element. The global stiffness matrix can be assembled so as to have the following form:

$$\mathbf{K} = \boldsymbol{\Phi}\boldsymbol{\Lambda}(\boldsymbol{\alpha})\boldsymbol{\Phi}^{\mathrm{T}}. \tag{20}$$

Suppose that the structure is composed of N_e elements, each of which has $N_{\alpha}^{(\mathrm{e})}$ unimodal components, connecting N_n nodes; every node has n_{dof} degrees of freedom. All the element parameters (cross-section properties and material properties of the elements) are collected in the vector $\boldsymbol{\alpha} = \begin{bmatrix} \boldsymbol{\alpha}^{(1)^{\mathrm{T}}} & \boldsymbol{\alpha}^{(2)^{\mathrm{T}}} & \dots & \boldsymbol{\alpha}^{(N^{(\mathrm{e})})^{\mathrm{T}}} \end{bmatrix}^{\mathrm{T}}$. The diagonal matrix $\boldsymbol{\Lambda}(\boldsymbol{\alpha})$

$$\boldsymbol{\Lambda}(\boldsymbol{\alpha}) = \mathrm{diag}[\boldsymbol{\alpha}] \tag{21}$$

lists the parameters of all the elements and its order $N_{\boldsymbol{\Lambda}}$ is equal to the number of unimodal components in the structure

$$N_{\boldsymbol{\Lambda}} = \sum_{e=1}^{N_e} N_{\alpha}^{(\mathrm{e})}. \tag{22}$$

The numerical matrix $\boldsymbol{\Phi}$ has dimension $(n_{\mathrm{dof}}N_n) \times N_{\Lambda}$ and it is built up by assembling the matrices $\boldsymbol{\Phi}^{(\mathrm{e})}$ according to element connectivity. Note that no overlapping occurs in this process, in fact $\boldsymbol{\Phi}$ is a partitioned matrix whose submatrix $[i,e]\,(i = 1,2,\dots,N_n;\ e = 1,2,\dots,N_e)$ has zero entries if the element e is not connected to node i, otherwise it is given by the n_{dof} rows of the matrix $\boldsymbol{\Phi}^{(\mathrm{e})}$ which are related to node i. The global stiffness matrix so obtained refers to a structure with no constraints, these can be included by modification of matrix $\boldsymbol{\Phi}$. For example, in order to impose a fixed degree of freedom one should delete the corresponding row of matrix $\boldsymbol{\Phi}$ (and the column of matrix $\boldsymbol{\Phi}^{\mathrm{T}}$). Once the boundary conditions have been imposed, one can see that the matrices $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^{\mathrm{T}}$ play the role of the statics and kinematics matrix, respectively. The diagonal matrix $\boldsymbol{\Lambda}(\boldsymbol{\alpha})$ rules the constitutive equations and is diagonal due to the particular choice of generalized stresses and strains.

In the literature, the global stiffness matrix is usually decomposed as in Eq. (20) just for truss structures. The extension to general structures with different finite elements, introduced by Fuchs (1991) and also employed by Elishakoff et al. (1997), Doebling et al. (1998) and by Hemez and Pagnacco (2000), was obtained by considering that any structure can be thought as an assemblage of unimodal elements in the same fashion of trusses.

4. Exact inverse of the stiffness matrix

The stiffness matrix given by Eq. (20) is an explicit function of the basic parameters collected in the vector $\boldsymbol{\alpha}$. If its parametric inverse, $\mathbf{K}(\boldsymbol{\alpha})^{-1}$, is available, then the explicit solution for the nodal displacements $\mathbf{u}(\boldsymbol{\alpha}) = \mathbf{K}(\boldsymbol{\alpha})^{-1}\mathbf{F}$, being $\mathbf{F}$ the vector of nodal forces, is easily evaluated. The explicit solution for other response quantities can be derived from the knowledge of the nodal displacements following standard finite element concepts. The exact inverse of the parametric stiffness matrix, which allows to express in explicit form the dependence of any structural response upon the basic variables, may be readily obtained for the following cases.

4.1. Statically determinate structures

For a statically determinate structure, i.e. a structure with a number of unknown displacements equal to the number of unimodal components ($n_{\mathrm{dof}}N_n = N_\Lambda$), $\mathbf{\Phi}$ is a square matrix. Then, the inverse of the stiffness matrix is simply given by

$$\mathbf{K}^{-1} = (\mathbf{\Phi}^{-1})^{\mathrm{T}}\mathbf{\Lambda}(\boldsymbol{\alpha})^{-1}\mathbf{\Phi}^{-1} \tag{23}$$

and reveals that the nodal displacements will be linear functions of the reciprocals of the basic parameters.

4.2. Structures with few redundancies

If the structure is statically indeterminate, there will be more unimodal components than unknown displacements, which lead to a rectangular matrix $\mathbf{\Phi}$ with the number of columns exceeding that of the rows (the difference between the number of columns, N_Λ, and rows, $n_{\mathrm{dof}}N_n$, equals the number of redundancies). After partitioning the matrices $\mathbf{\Phi}$ and $\mathbf{\Lambda}(\boldsymbol{\alpha})$, the stiffness matrix can be split in two matrices

$$\mathbf{K} = [\,\mathbf{\Phi}_1 \quad \mathbf{\Phi}_2\,]\begin{bmatrix} \mathbf{\Lambda}_1(\boldsymbol{\alpha}_1) & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_2(\boldsymbol{\alpha}_2) \end{bmatrix}\begin{bmatrix} \mathbf{\Phi}_1^{\mathrm{T}} \\ \mathbf{\Phi}_2^{\mathrm{T}} \end{bmatrix} = \mathbf{K}_1 + \mathbf{\Phi}_2\mathbf{\Lambda}_2(\boldsymbol{\alpha}_2)\mathbf{\Phi}_2^{\mathrm{T}} \tag{24}$$

where

$$\mathbf{K}_1 = \mathbf{\Phi}_1\mathbf{\Lambda}_1(\boldsymbol{\alpha}_1)\mathbf{\Phi}_1^{\mathrm{T}}. \tag{25}$$

When the stiffness matrix is expressed by Eq. (24), its inverse can be derived by the use of the SMW formula (Henderson and Searle, 1981; Hager, 1989), which gives

$$\mathbf{K}^{-1} = \mathbf{K}_1^{-1} - \mathbf{K}_1^{-1}\mathbf{\Phi}_2(\mathbf{\Lambda}_2^{-1}(\boldsymbol{\alpha}_2) + \mathbf{\Phi}_2^{\mathrm{T}}\mathbf{K}_1^{-1}\mathbf{\Phi}_2)^{-1}\mathbf{\Phi}_2^{\mathrm{T}}\mathbf{K}_1^{-1}. \tag{26}$$

The previous relationship is of practical interest when one can easily derive the inverse of the matrices $\mathbf{K}_1$ and $(\mathbf{\Lambda}_2^{-1}(\boldsymbol{\alpha}_2) + \mathbf{\Phi}_2^{\mathrm{T}}\mathbf{K}_1^{-1}\mathbf{\Phi}_2)$, which happens, for example, if the structure has few redundancies. In this case, the partition of the stiffness matrix should be performed so that $\mathbf{\Phi}_1$ is the square statics matrix of any statically determinate substructure, with parameters $\boldsymbol{\alpha}_1$, extracted by the original structure. Then, $\mathbf{K}_1$ is the stiffness matrix of a statically determinate substructure and its inverse is obtained by Eq. (23)

$$\mathbf{K}_1^{-1} = (\mathbf{\Phi}_1^{-1})^{\mathrm{T}}\mathbf{\Lambda}_1^{-1}(\boldsymbol{\alpha}_1)\mathbf{\Phi}_1^{-1} \tag{27}$$

The parametric matrix $(\mathbf{\Lambda}_2^{-1}(\boldsymbol{\alpha}_2) + \mathbf{\Phi}_2^{\mathrm{T}}\mathbf{K}_1^{-1}\mathbf{\Phi}_2)$, which includes the statics matrix $\mathbf{\Phi}_2$ of the redundant elements and their parameters $\boldsymbol{\alpha}_2$, has dimension equal to the number of redundancies and can be inverted in explicit form due to its small dimension.

Eq. (26) clearly shows that the explicit solutions related to statically indeterminate structures will contain cross-terms of the basic variables, which account for the coupling of the effects of parameter variations on the response. Such terms are missing in the explicit solution pertaining to statically determinate structures.

4.3. Structures with few parameters

Eq. (26) also allows to derive the exact inverse of the parametric stiffness matrix for the case in which the properties of just few elements are let to vary, i.e. few parameters are present. For this scenario the partition is performed so as to confine the parameters of the variable unimodal components inside the matrix $\mathbf{\Lambda}_2(\boldsymbol{\alpha}_2)$. Then, $\mathbf{K}_1$ is the stiffness matrix of the structure deprived of the parametric unimodal components, whose inversion is obtained numerically (being a non-parametric matrix). The inverse of the parametric matrix $(\mathbf{\Lambda}_2^{-1}(\boldsymbol{\alpha}_2) + \mathbf{\Phi}_2^{\mathrm{T}}\mathbf{K}_1^{-1}\mathbf{\Phi}_2)$ can be derived in explicit form due to its small dimensions; for example, if the elastic modulus of a beam element of the structure is treated as a parameter, then that matrix will be of order 3.

5. Approximate inverse of the stiffness matrix

The procedure previously described exploits the effectiveness of the SMW formula to treat low-rank modifications but it is not suitable for the general case of structures with many redundancies and many parameters (which are related to high-rank modifications). Anyway, the availability of the exact inverse of the stiffness matrix for this kind of problems has a scarce practical relevance. In fact, when redundancies and parameters increase together, the exact inverse would be difficult to handle being represented by very lengthy analytical expressions. Of more interest is to determine an accurate approximate inverse of the stiffness matrix. An effective procedure to achieve this aim is proposed in the following.

Assume first that only the ith unimodal component is let to vary while all the others have fixed values. The parameter α_i will be decomposed as

$$\alpha_i = \alpha_{0_i} + \lambda_i \tag{28}$$

where α_{0i} is the reference value and λ_i is the fluctuation around the reference value. Denote with $\mathbf{K}_0$ the reference stiffness matrix (where the parameter takes its reference value). Then, the contribution of the unimodal component fluctuation to the stiffness matrix is $\mathbf{K}_i = \lambda_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}}$ (where $\boldsymbol{\phi}^i$ is the ith column of $\boldsymbol{\Phi}$) and the SMW formula gives

$$\mathbf{K}^{-1} = (\mathbf{K}_0 + \lambda_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}})^{-1} = \mathbf{K}_0^{-1} - \frac{\lambda_i}{1 + \lambda_i a_i} \mathbf{A}_i \tag{29}$$

where the following numerical quantities appear:

$$a_i = \boldsymbol{\phi}_i^{\mathrm{T}} \mathbf{K}_0^{-1} \boldsymbol{\phi}_i; \quad \mathbf{A}_i = \mathbf{K}_0^{-1} \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}} \mathbf{K}_0^{-1} \tag{30}$$

Of course the choice of the reference value α_{0i} is of no importance in the evaluation of the parametric matrix $\mathbf{K}^{-1}$, being Eq. (29) an exact relationship.

For the general case in which N parametric unimodal components are present ($N \leqslant N_\Lambda$), one can assume as a first approximation that the contributions of the fluctuations are linearly superimposed so that the approximate inverse reads

$$\mathbf{K}^{-1} = \left(\mathbf{K}_0 + \sum_{i=1}^{N} \lambda_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}} \right)^{-1} \approx \mathbf{K}_0^{-1} - \sum_{i=1}^{N} \frac{\lambda_i}{1 + \lambda_i a_i} \mathbf{A}_i \tag{31}$$

where the reference stiffness matrix is given by

$$\mathbf{K}_0 = \boldsymbol{\Phi} \boldsymbol{\Lambda}(\boldsymbol{\alpha}_0) \boldsymbol{\Phi}^{\mathrm{T}}; \quad \boldsymbol{\alpha}_0 = [\alpha_{01} \quad \alpha_{02} \quad \cdots \quad \alpha_{0N}]^{\mathrm{T}} \tag{32}$$

Eq. (31) states that the approximate inverse is obtained by superimposing the contributions due to parameter fluctuations to the inverse of the reference stiffness matrix. Each contribution is expressed by the product of the matrix $\mathbf{A}_i$ and the rational function $\lambda_i/(1 + \lambda_i \alpha_i)$. As regard the choice of the reference values α_{0i}, it should be noted that it would influence the analytical solution produced by Eq. (31), being this one an approximate relationship. One could set the reference values α_{0i} equal to the initial design values of the parameters α_i^*. However, this is adequate only if the mean value of the ratio between the parameter fluctuations around the initial design value, $\lambda_i^* = \alpha_i - \alpha_i^*$, and the initial design value itself

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \frac{\lambda_i^*}{\alpha_i^*} \tag{33}$$

is small, i.e. when the parameter fluctuations are evenly spread around their initial design values. A better choice for the reference values of the parameters, which removes this limitation, is to assume $\alpha_{0i} = (1 + \mu)\alpha_i^*$. So operating, in Eqs. (30) and (31)

$$\mathbf{K}_0^{-1} = \frac{1}{(1 + \mu)} \mathbf{K}^{*-1}; \quad \lambda_i = \lambda_i^* - \mu \alpha_i^* \tag{34}$$

being $\mathbf{K}^* = \boldsymbol{\Phi} \boldsymbol{\Lambda}(\boldsymbol{\alpha}^*) \boldsymbol{\Phi}^{\mathrm{T}}$ the stiffness matrix of the initial design structure. Note that when Eq. (34) is employed, Eq. (31) gives the exact inverse for modified structures obtained by just scaling the initial design stiffness matrix, so that $\mathbf{K} = \gamma \mathbf{K}^*$.

Eq. (31) produces accurate solutions if the fluctuations λ_i are not too broad. This is because it lacks the cross-terms which should account for the interaction effects among fluctuations of different unimodal components. A significant improvement of Eq. (31) can be achieved by including just the cross-terms relative to the couples of unimodal parameter fluctuations. To this aim, let us assume that only a couple of parametric unimodal components is present: α_{0i} and α_{0j} are the reference values whereas λ_i and λ_j are the fluctuations. In this case, by applying the SMW formula twice, recursively, one gets

$$\begin{aligned}\mathbf{K}^{-1} &= (\mathbf{K}_0 + \boldsymbol{\lambda}_i\boldsymbol{\phi}_i\boldsymbol{\phi}_i^{\mathrm{T}} + \boldsymbol{\lambda}_j\boldsymbol{\phi}_j\boldsymbol{\phi}_j^{\mathrm{T}})^{-1} \\ &= \mathbf{K}_0^{-1} - \frac{\lambda_i}{1+\lambda_i a_i}\mathbf{A}_i - \frac{\lambda_j}{1+\lambda_j a_j}\mathbf{A}_j \\ &\quad - \frac{\lambda_i\lambda_j a_{ij}^2}{1+\lambda_i a_i + \lambda_j a_j + \lambda_i\lambda_j(a_i a_j - a_{ij}^2)}\left[\frac{\lambda_i}{1+\lambda_i a_i}\mathbf{A}_i + \frac{\lambda_j}{1+\lambda_j a_j}\mathbf{A}_j - \frac{1}{a_{ij}}(\mathbf{A}_{ij} + \mathbf{A}_{ij}^{\mathrm{T}})\right]\end{aligned} \tag{35}$$

where the new numerical quantities, a_{ij} and $\mathbf{A}_{ij}$, read

$$a_{ij} = \boldsymbol{\phi}_i^{\mathrm{T}}\mathbf{K}_0^{-1}\boldsymbol{\phi}_j; \quad \mathbf{A}_{ij} = \mathbf{K}_0^{-1}\boldsymbol{\phi}_i\boldsymbol{\phi}_j^{\mathrm{T}}\mathbf{K}_0^{-1} \tag{36}$$

If N parametric unimodal components are present, they can be combined into $N(N-1)/2$ couples. Then, according to Eq. (35), the approximate inverse of the stiffness matrix, which accounts for the effects due to the simultaneous variation of all parameter couples, reads

$$\begin{aligned}\mathbf{K}^{-1} &= \left(\mathbf{K}_0 + \sum_{i=1}^{N}\lambda_i\boldsymbol{\phi}_i\boldsymbol{\phi}_i^{\mathrm{T}}\right)^{-1} \approx \mathbf{K}_0^{-1} - \sum_{i=1}^{N}\frac{\lambda_i}{1+\lambda_i a_i}\mathbf{A}_i \\ &\quad - \sum_{\substack{i=1\\ j=i+1}}^{N}\frac{\lambda_i\lambda_j a_{ij}^2}{1+\lambda_i a_i + \lambda_j a_j + \lambda_i\lambda_j(a_i a_j - a_{ij}^2)}\left[\frac{\lambda_i}{1+\lambda_i a_i}\mathbf{A}_i + \frac{\lambda_j}{1+\lambda_j a_j}\mathbf{A}_j - \frac{1}{a_{ij}}(\mathbf{A}_{ij} + \mathbf{A}_{ij}^{\mathrm{T}})\right]\end{aligned} \tag{37}$$

where the relationships in Eq. (34) should be employed.

Eq. (37) is close to the exact inverse of the parametric stiffness matrix even for large parameter fluctuations. Indeed, Eq. (37) differs from the exact inverse only for the absence of the terms accounting for the effects of coupling among three or more unimodal components, which are small even for large deviations from the initial design values.

The number of terms appearing into Eq. (37) can be considerably reduced by performing a sensitivity analysis. Indeed, many of the cross-terms are negligible in comparison to the others and can be dropped from the sum. Keeping low the number of terms increases the efficiency of the proposed formulation especially when the approximate explicit inverse is exploited in the framework of reanalysis or reliability analysis in conjunction with digital simulations to evaluate the response statistics of interest.

6. Applications

The approximate inverse of the stiffness matrix leads to the sought approximate explicit solutions in terms of nodal displacements, which allow to determine explicit expressions for other response quantities. In the following, the proposed formulation is applied to a truss structure, a frame structure and a 2D structure modeled by constant strain triangles. The element stiffness matrix is formed according to paragraphs (a), (b) and (c) of Section 2, then the global stiffness matrix is assembled by the technique described in Section 3. Note that such a procedure is very general and can be easily implemented into computer codes. Once the numerical inverse of the initial design stiffness matrix $\mathbf{K}^{*-1}$ has been computed, the approximate inverse of the parametric stiffness matrix is determined by both Eqs. (31) and (37), in each case making use of Eq. (34). The accuracy of the proposed approximate explicit solutions is tested by computing the error with respect to exact numerical solution obtained by fresh analysis. Specifically, the error of the generic quantity q (nodal displacement or internal stress) is defined as

$$\mathscr{E}(q) = \frac{q_{\mathrm{exact}} - q_{\mathrm{approximate}}}{q_{\mathrm{approximate}}} \times 100 \tag{38}$$

6.1. Truss structure

The approximate explicit relationship between nodal displacements and element parameters $\alpha_i = 2E^{(i)}A^{(i)}/l^{(i)}$ ($i = 1, 2, \ldots, 24$) has been evaluated for the truss structure with six redundancies shown in Fig. 2. The initial design value for the element parameters is $\alpha_i^* = 2E^*A^*/l^{(i)}$, being E^*A^* the axial stiffness of all the elements in the initial design structure.

Table 1 lists the percentage error of the horizontal displacement at the upper-right node, u, for some sample structures obtained by varying the value of $E^{(i)}A^{(i)}$ in the elements. The results obtained by Eq. (37) are very accurate even for structures which are quite different from the initial design one (where $E^{(i)}A^{(i)} = E^*A^*$), due to large modification of the parameters. The error pertaining to Eq. (31) is always larger than that related to Eq. (37), thus testifying the important role played by cross-terms.

The contribution of cross-terms in the approximate explicit solution for the displacement u, given by the double sum in Eq. (37), is shown in Fig. 3. The figure illustrates the sensitivity of u with respect to parameter couples, which is defined as

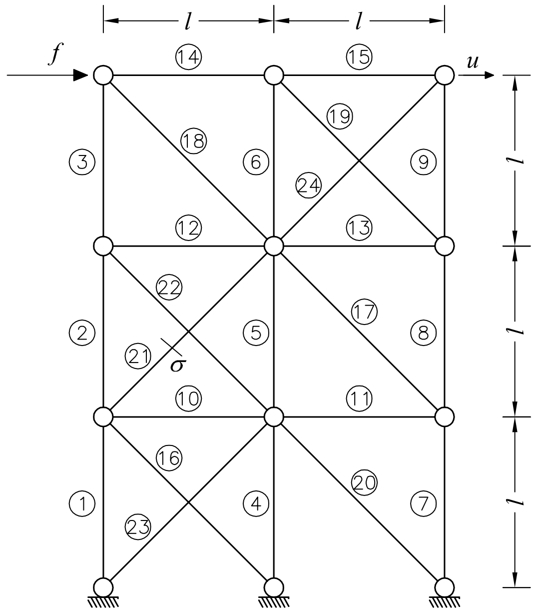

Fig. 2. Truss structure.

Table 1
Exact solution and percentage error $\mathscr{E}(u)$ of the approximate solutions of the horizontal displacement at the upper-right node for the initial design truss and for some sample structures

	$E^{(i)}A^{(i)}/E^*A^*$ in the elements				$u_{\text{exact}} \times \frac{E^*A^*}{fl}$	$\mathscr{E}(u)$ (Eq. (31))	$\mathscr{E}(u)$ (Eq. (37))
	From element 1 to 6	From element 7 to 12	From element 13 to 18	From element 19 to 24			
Initial design	1	1	1	1	6.42	0	0
Samples	1.3	1.3	1.3	0.7	6.19	0.67	−0.22
	2	2	2	0.5	5.27	3.8	−2.6
	0.5	0.5	1.3	1.3	9.22	3.42	0.01
	0.3	0.3	1.5	1.5	13.32	11.25	0.48
	1.5	1.5	0.5	0.5	7.85	8.93	1.54
	2	0.5	0.5	0.5	9.27	1.21	0.60

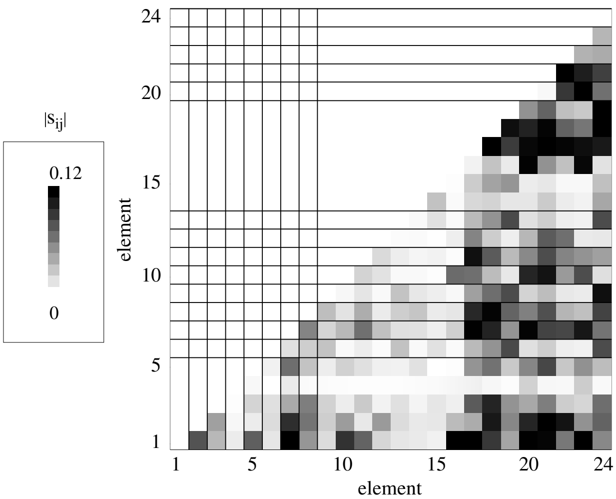

Fig. 3. Absolute value of the sensitivity s_{ij} (Eq. (39)) of the approximate displacement u of the truss structure with respect to element parameter couples.

$$s_{ij} = \frac{E^*A^*}{fl^3}\frac{\partial u}{\partial\alpha_i\,\partial\alpha_j}\bigg|_{\substack{\alpha_i=\alpha_i^*\\ \alpha_j=\alpha_j^*}} \tag{39}$$

The main coupling effects are given by the element couples depicted by dark grey in Fig. 3. The major contribution is provided by the interaction between elements 20 and 23.

Fig. 4 displays the probability density function of the absolute error $|\mathscr{E}(u)|$ when each element stiffness is independently randomly chosen in the range $[0.33E^*A^*, 3E^*A^*]$. This function has been obtained by solving a large number of sample structures. In each of them the axial stiffness of the generic bar is the realization of an independent random variable uniformly distributed in the range $[0.33E^*A^*, 3E^*A^*]$. Then, a sample of the error associated to each structure is computed. Fig. 4 gives information on the probability that the error exceeds a given value when the approximate explicit relationships are employed. As clear in the figure, despite the broad range of fluctuation, the probability of encountering an error greater than 2% is small when Eq. (37)

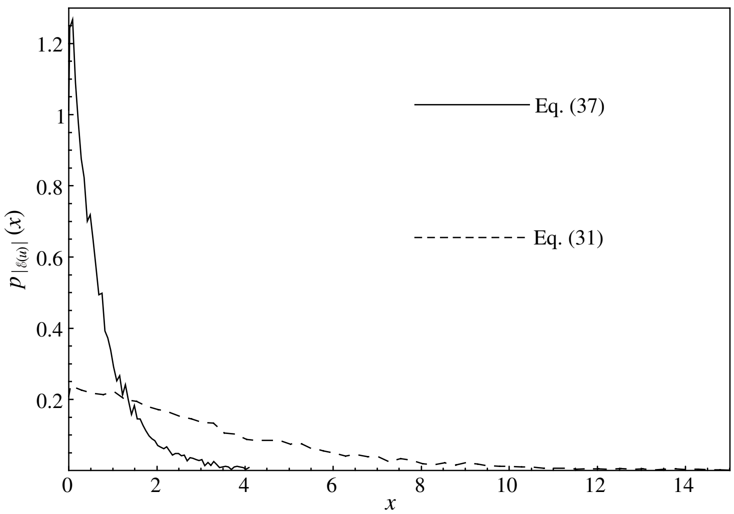

Fig. 4. PDF of the absolute value of the percentage error $|\mathscr{E}(u)|$ for the displacement u of the truss structure when each element stiffness is independently randomly chosen in the range $[0.33E^*A^*, 3E^*A^*]$.

is adopted and the chance to find an error greater than 4% is almost zero. The approximate solution pertaining to Eq. (31) is less accurate, as errors up to 15% are produced although values greater than 10% are encountered with small probability.

The probability density function of the absolute error $|\mathscr{E}(\sigma)|$, being σ the axial stress of bar 21, is plotted in Fig. 5. It can be seen that the approximate solution based on Eq. (37) leads to remarkable accuracy also when

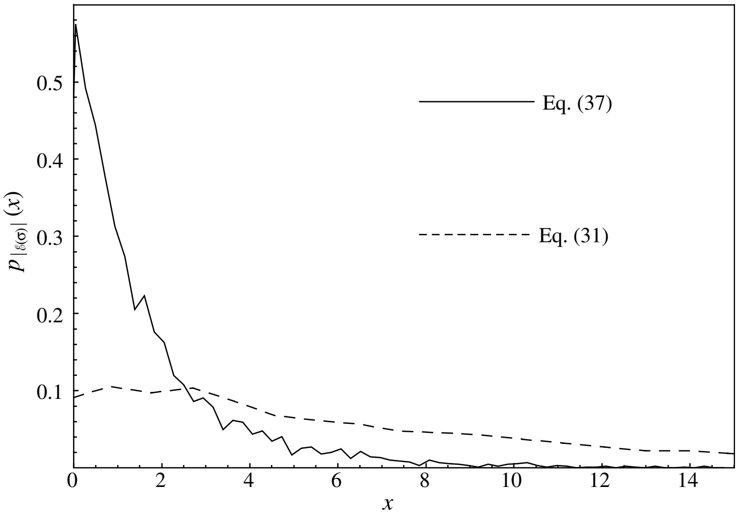

Fig. 5. PDF of the absolute value of the percentage error $|\mathscr{E}(\sigma)|$ for the axial stress σ of bar 21 of the truss structure when each element stiffness is independently randomly chosen in the range $[0.33E^{*}A^{*}, 3E^{*}A^{*}]$.

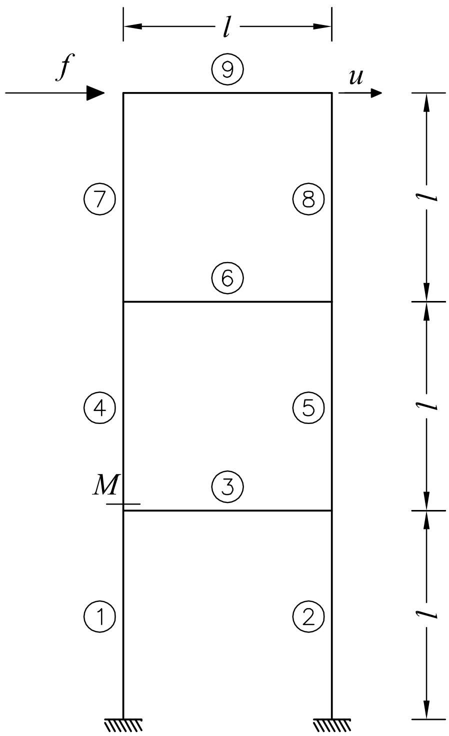

Fig. 6. Frame structure.

internal stresses are of concern. In this case, in fact, as shown in the figure, the probability to produce an error greater than 10% is basically zero. On the contrary, a larger error is encountered if the cross-terms are neglected according to Eq. (31).

6.2. Frame structure

The frame structure reported in Fig. 6 has been analyzed. The structure, composed of $N_e = 9$ Euler elements, has $n_{\text{dof}} \times N_n = 18$ degrees of freedom, so that $N_\Lambda = 27$ unimodal components are present (the redundancies are $N_\Lambda - n_{\text{dof}} \times N_n = 9$). The axial and bending stiffness of all the elements of the initial design structure are $E^* A^*$ and $E^* I^* = E^* A^* \times (l/100)^2$, respectively.

Table 2 shows the error of the explicit solutions for the horizontal displacement at the upper-right node, u, for some sample structures with different values of $E^{(e)}A^{(e)}$ and $E^{(e)}I^{(e)}$ in the elements. The error related to Eq. (37) is very small for all parent structures.

Fig. 7 depicts the absolute error probability density function for the same displacement component when the values of $E^{(e)}A^{(e)}$ and $E^{(e)}I^{(e)}$ of each element are independently randomly chosen in the ranges $[0.33E^*A^*, 3E^*A^*]$ and $[0.33E^*I^*, 3E^*I^*]$, respectively.

Table 2
Exact solution and percentage error $\mathscr{E}(u)$ of the approximate solutions of the horizontal displacement at the upper-right node for the initial design frame and for some sample structures

	$E^{(e)}A^{(e)}/E^*A^*$ in the elements		$E^{(e)}I^{(e)}/E^*I^*$ in the elements		$u_{\text{exact}} \times \frac{E^*I^*}{fl^3}$	$\mathscr{E}(u)$ (Eq. (31))	$\mathscr{E}(u)$ (Eq. (37))
	From element 1 to 5	From element 6 to 9	From element 1 to 5	From element 6 to 9			
Initial design	1	1	1	1	0.299	0	0
Samples	1	1	2	2	0.151	1.75	0.17
	2	1	1	2	0.225	2.13	0.21
	1	2	2	1	0.217	1.13	−0.50
	2	0.5	0.5	2	0.373	11.63	1.03
	2	0.5	0.5	0.5	0.597	9.85	1.02
	1	1	0.3	1	0.648	8.22	0.75

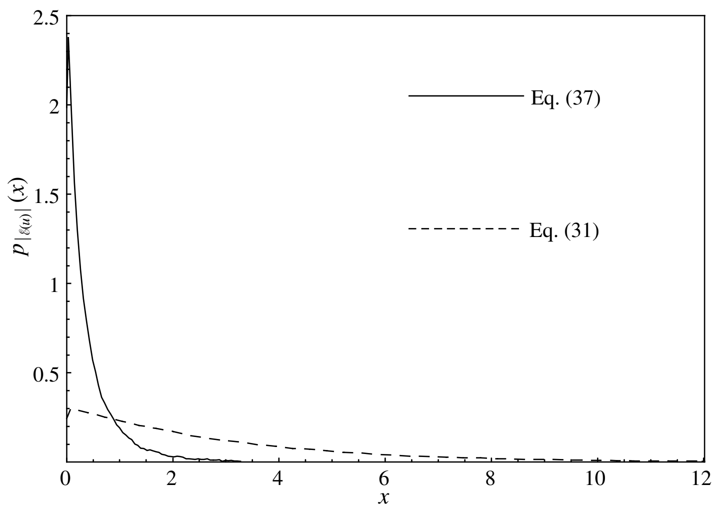

Fig. 7. PDF of the absolute value of the percentage error $|\mathscr{E}(u)|$ for the horizontal displacement at the upper-right node of the frame structure when the axial and bending stiffness of each element are independently randomly chosen in the ranges $[0.33E^*A^*, 3E^*A^*]$ and $[0.33E^*I^*, 3E^*I^*]$.

The probability density function of the absolute error $|\mathscr{E}(M)|$ for the bending moment M at the base of element 4 is plotted in Fig. 8.

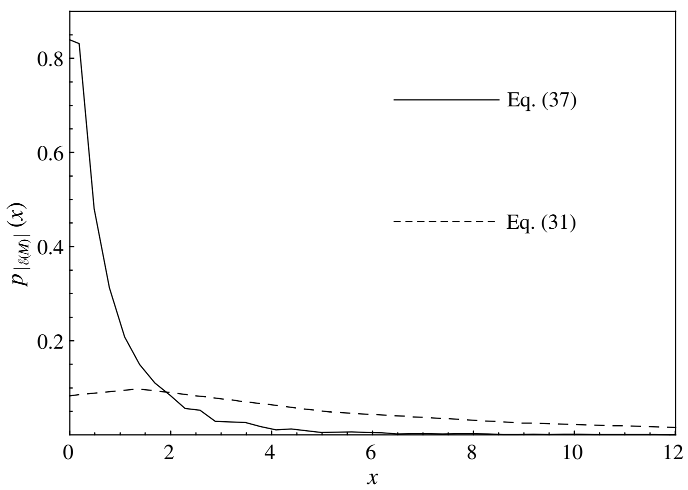

Fig. 8. PDF of the absolute value of the percentage error $|\mathscr{E}(M)|$ for the bending moment at the base of element 4 of the frame structure when the axial and bending stiffness of each element are independently randomly chosen in the ranges $[0.33E^*A^*, 3E^*A^*]$ and $[0.33E^*I^*, 3E^*I^*]$.

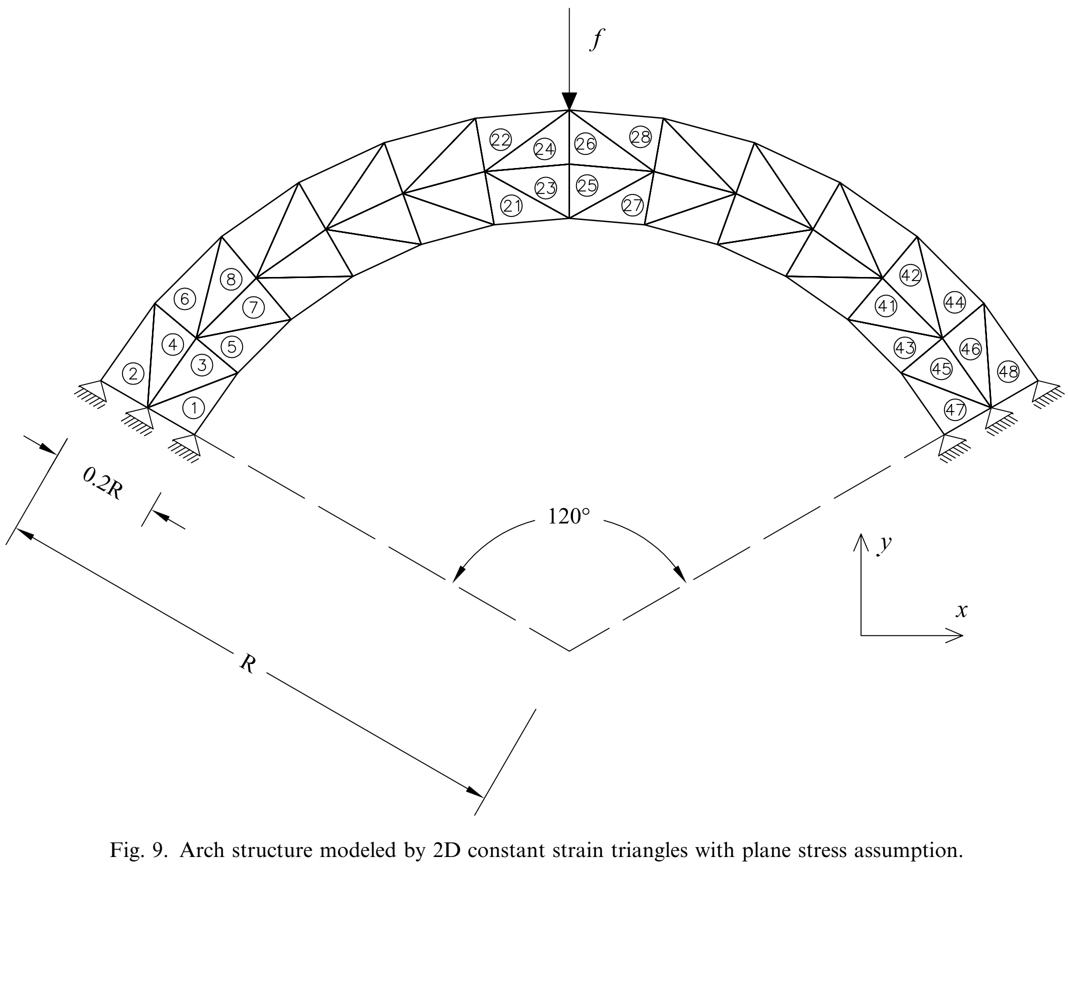

Fig. 9. Arch structure modeled by 2D constant strain triangles with plane stress assumption.

Figs. 7 and 8 demonstrate the effectiveness of the proposed approximate explicit solutions also for framed structures. In fact, both for displacements and internal stresses, it is reproduced the same level of accuracy encountered for the truss of the previous example. The better performance of the explicit solution with cross-terms is assessed by the lower probability to exceed a given error threshold. Furthermore, the comparison between Figs. 7 and 8 clearly shows that almost the same accuracy level is associated with displacement and stress components.

6.3. 2D structure modeled by constant strain triangles

The plane stress problem given by the arch modeled by constant strain triangles shown in Fig. 9 has been analyzed assuming a fixed value of the Poisson ratio $\nu = 0.3$. The structure, composed of $N_e = 48$ elements, has $n_{\text{dof}} \times N_n = 66$ degrees of freedom, so that $N_\Lambda = 144$ unimodal components are present (the redundancies are $N_\Lambda - n_{\text{dof}} \times N_n = 78$).

The percentage error $\mathscr{E}(v)$ pertaining to the approximate explicit expression of the vertical displacement of the loaded node, v (see Fig. 9) is reported in Table 3.

Fig. 10 portrays the probability density function of the absolute error $|\mathscr{E}(v)|$, for the same displacement component, when the value of the stiffness $E^{(e)}t^{(e)}$ of each element is randomly chosen in the range $[0.33E^*t^*, 3E^*t^*]$ (being E^* and t^* the properties of all the elements of the initial design structure).

Table 3
Exact solution and percentage error $\mathscr{E}(v)$ of the approximate solutions of the vertical displacement of the loaded node for the initial design arch and for some sample structures

	$E^{(e)}t^{(e)}/E^*t^*$ in the elements				$v_{\text{exact}} \times \frac{E^*t^*}{f}$	$\mathscr{E}(v)$ (Eq. (31))	$\mathscr{E}(v)$ (Eq. (37))
	From element 1 to 12	From element 13 to 24	From element 25 to 36	From element 37 to 48			
Initial design	1	1	1	1	0.092	0	0
Samples	1.5	0.7	0.7	1.5	0.103	5.75	0.37
	2	2	2	0.7	0.060	7.17	0.93
	1.5	1.5	0.7	1.5	0.082	5.84	0.42
	0.7	0.7	1.5	1.5	0.095	5.39	0.11
	0.5	0.5	1.5	1.5	0.117	11.23	0.57
	2	0.5	0.5	0.5	0.152	10.92	0.50

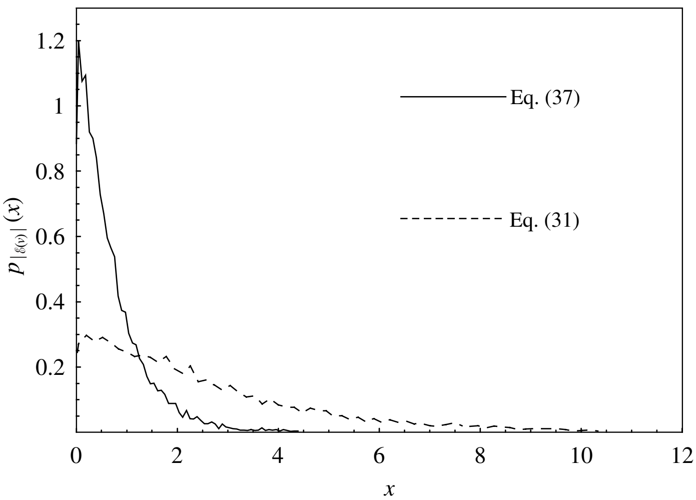

Fig. 10. PDF of the absolute value of the percentage error $|\mathscr{E}(v)|$ for the vertical displacement of the loaded node of the arch structure when the stiffness of each element is independently randomly chosen in the range $[0.33E^*t^*, 3E^*t^*]$.

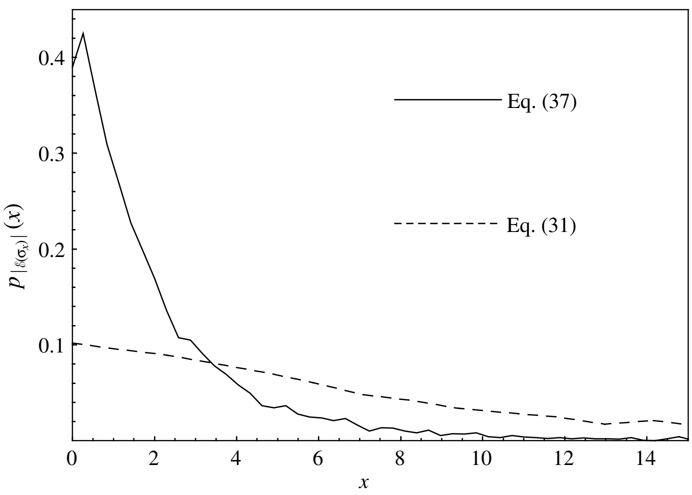

Fig. 11. PDF of the absolute value of the percentage error $|\mathscr{E}(\sigma_x)|$ for the stress σ_x in element 46 of the arch structure when the stiffness of each element is independently randomly chosen in the range $[0.33E^*t^*, 3E^*t^*]$.

The probability density function of the absolute error $|\mathscr{E}(\sigma_x)|$, affecting the normal stress σ_x in element 46, is shown in Fig. 11.

As evident from Table 3, Figs. 10 and 11, although the number of unimodal components and redundancies is much larger than in the two previous applications, the same accuracy level is achieved (compare Figs. 4, 7 and 10 and Figs. 5, 8 and 11). In particular, the approximate explicit expression which hinges on Eq. (37) appears very effective for predicting displacements and internal stresses of finite element modeled structures.

7. Conclusions

The paper has presented a procedure for evaluating, via inversion of the stiffness matrix, an approximate explicit relationship between the static nodal displacements of linear finite element modeled structures and the element properties. The technique is based on a non-conventional assemblage of the stiffness matrix which is made possible thanks to the factorization of the element stiffness matrix according to the unimodal components concept. Basically, each finite element is decomposed into its unimodal components, i.e. sub-elements acting in parallel with the same connectivity of the original element which are assembled in a truss-like fashion. So operating, the exact inverse of the global stiffness matrix, in which the element properties appear as parameters, can be easily obtained for the case of statically determinate structures (i.e. structures with a number of unimodal components equal to the number of degrees of freedom). The exact inverse is also effectively provided for structures with few redundancies or few parametric element properties by making use of the Sherman–Morrison–Woodbury formula. For the general setting of structures with many redundancies and parameters the exact explicit solution is still achievable in principle, however the analytical relationships between the response and the basic parameters would be practically unusable because of the immense number of terms involved. Then, two approximate solutions have been proposed which retain the most significant terms. The first, which simply superimposes the contribution of every unimodal component fluctuation, is a crude one and should be used only if the parameters have small variability ranges. The second approximate solution takes into account also the interaction effects between unimodal component couples so to produce very accurate results, both in terms of displacements and internal stresses, even for broad parameter fluctuations. The numerical applications have shown that the accuracy does not deteriorate when the redundancies and the degrees of freedom increase whatever type of finite element is used. Furthermore, the proposed technique can be easily implemented into standard finite element codes and produces explicit solutions which can be handled for further processing, so that it can be conveniently exploited for reanalysis, optimization, identification, inverse analysis, reliability analysis, etc.

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References

Abu Kassim, A.M., Topping, B.H.V., 1987. Static reanalysis: a review. J. Struct. Eng. ASCE 113, 1029–1045.
Akgün, M.A., Garcelon, J.H., Haftka, T.R., 2001. Fast exact linear and non-linear structural reanalysis and the Sherman–Morrison–Woodbury formulas. Int. J. Numer. Meth. Eng. 50, 1587–1606.
Di Paola, M., 2004. Probabilistic analysis of truss structures with uncertain parameters (virtual distortion method approach). Probabilist. Eng. Mech. 19, 321–329.
Doebling, S.W., Peterson, L.D., Alvin, K.F., 1998. Experimental determination of local structural stiffness by disassembly of measured flexibility matrices. J. Vib. Acoust. 120, 949–957.
Elishakoff, I., Ren, Y.J., 2003. Finite Element Methods for Structures With Large Stochastic Variations. Oxford University Press, Oxford.
Elishakoff, I., Ren, Y.J., Shinozuka, M., 1997. New formulation of FEM for deterministic and stochastic beams through generalization of Fuchs' method. Comput. Methods Appl. Mech. Eng. 144, 135–143.
Falsone, G., Impollonia, N., 2002. A new approach for the stochastic analysis of finite element modeled structures with uncertain parameters. Comput. Methods Appl. Mech. Eng. 191, 5067–5085.
Falsone, G., Impollonia, N., 2004. About the accuracy of a novel response surface method for the analysis of finite element modeled uncertain structures. Probabilist. Eng. Mech. 19, 53–63.
Falsone, G., Sofi, A., 2004. Exact solutions for statically indeterminate stochastic beams. In: Proceedings of the 7th International Conference on Computational Structures Technology, Lisbon.
Fuchs, M.B., 1991. Unimodal beam elements. Int. J. Solids Struct. 27, 533–545.
Fuchs, M.B., 1992a. The explicit inverse of the stiffness matrix. Int. J. Solids Struct. 29, 2101–2113.
Fuchs, M.B., 1992b. Analytical representation of member forces in linear elastic redundant trusses. Int. J. Solids Struct. 29, 519–530.
Fuchs, M.B., 1997. Unimodal formulation of the analysis and design problems for framed structures. Comput. Struct. 63, 739–747.
Fuchs, M.B., Maslovitz, G., 1992. Approximate reanalysis based on the exact analytical expressions. AIAA J. 30, 1632–1637.
Fuchs, M.B., Shabtay, E., 2000. The reciprocal approximation in stochastic analysis of structures. Chaos, Solitons Fractals 11, 889–900.
Garcelon, J.H., Haftka, R.T., Scotti, S.J., 2000. Approximations in optimization of damage tolerant structures. AIAA J. 38, 517–524.
Hager, W.W., 1989. Updating the inverse of a matrix. SIAM Rev. 31, 221–239.
Hemez, F.M., Pagnacco, E., 2000. Statics and inverse dynamics solvers based on strain-mode disassembly. Eur. J. Finite Elem. 9, 511–560.
Henderson, H.V., Searle, S.R., 1981. On deriving the inverse of a sum of matrices. SIAM Rev. 23, 53–60.
Kavlie, D., Powell, G.H., 1971. Efficient reanalysis of modified structures. J. Struct. Div. ASCE 97 (ST1), 377–392.
Kirsch, U., Liu, S., 1995. Exact structural reanalysis by a first-order reduced basis approach. Struct. Optim. 10, 153–158.
Rong, T.-Y., Lü, A.-Q., 1994. Theory and method of structural variations of finite element systems. AIAA J. 32, 1911–1919.
Schmit, L.A., Farhi, B., 1974. Some approximation concepts for structural synthesis. AIAA J. 18, 1515–1524.
Topping, B.H.V., Abu Kassim, A.M., 1987. The use and efficiency of the theorems of structural variation for finite element analysis. Int. J. Numer. Methods Eng. 24, 1901–1920.